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A charged particle in a time-varying magnetic field

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Abstract. Electron interaction with a uniform magnetic field, suddenly (by leaps) changing in time, is considered theoretically. Characteristic features of the switching-on process are derived; in particular, it is shown that at low values E/B (E is the energy of the electron before switching, B is the intensity of the magnetic field) transitions are possible only to even Landau states, and at large E/B there is an interaction only with high-lying levels. Generalizations to more than one switching are presented. If the magnetic field is switched off and switched on again after time T, then electrons from an even (odd) Landau state can make transitions only to an even (odd) level. Varying the ratio ω_B/ω (ω_B is the cyclotron frequency, $\omega=\pi/T$), one can efficiently control the probabilities of electromagnetic emission or absorption. Various results of calculations of the switching process from $B_1\neq 0$ to $B_2\neq 0$ are also presented. In the appendix some properties of the functions $H_n^*(x)=(-i)^n H_n(ix)$ are given.

Attainability of strong magnetic fields is raising the problem of an adequate theoretical description of their influence on microstructures. It is well known that in the general case of a time-dependent magnetic field it is impossible to separate time and space variables in the Schrödinger equation. Thus one utilizes miscellaneous assumptions and auxiliary methods. One such approach is used in this paper, where, on the basis of exact quantum-mechanical analysis, charged particle interaction with a magnetic field, suddenly (by leaps) changing in time, is considered theoretically. To the surprise of the author, no solution of this simple but instructive problem has been found in the literature. It follows from Maxwell's equations that in this case an electric field is induced which is not zero at the switching moment only. This electric field, having as a function of time the form of one, or several, δ-functions, changes the energy spectrum of the particle, e.g. it can transform from a continuous to discrete spectrum and vice versa. In the process of transition from one state to another, the electron will emit or absorb electromagnetic radiation. We will show that changing the switching frequency or (and) magnetic field intensity, one can efficiently control this process.

Let us start by considering the situation when the magnetic field $\boldsymbol{B}=(0,0,B)$ is suddenly switching on at the moment $t=0$: $\boldsymbol{B}(t)=\boldsymbol{B}h(t)$, where $h(t)$ is the Heaviside step function. We choose the vector potential in Landau gauge $\boldsymbol{A}=(-By,0,0)$. At $t<0$ the particle is described by a δ-normalized plane wave:

$$\psi(x,y,z,t)=\frac{1}{(2\pi\hbar)^{3/2}}\exp\left\{\frac{i}{\hbar}\left[p_x x+p_z z+p_y(y-y_0)\right]\right\}\times\exp\left[-i\frac{p_x^2+p_y^2+p_z^2}{2m_p\hbar}t\right] \qquad (1)$$

where the auxiliary phase factor $\exp[-(i/\hbar)p_y y_0]$ ($y_0=-p_x/eB$, the centre of magnetic oscillations) is introduced for convenience of further calculations. At $t>0$ one has superposition of the usual Landau states (magnetic field does not change x and z components of kinetic momentum p, thus wavefunction dependence on these variables remains the same):

$$\psi(x,y,z,t)=\frac{1}{2\pi\hbar}\exp\left\{\frac{i}{\hbar}[p_x x+p_z z]\right\}\exp\left(-i\frac{p_z^2}{2m_p\hbar}t\right)$$
$$\times\sum_{n=0}^{\infty}C_n(p_y)\exp\left[-i\left(n+\frac{1}{2}\right)\omega_B t\right]\chi_n(y) \qquad (2)$$

$$\chi_n(y)=\frac{1}{\pi^{1/4}r_B^{1/2}\sqrt{2^n n!}}\exp\left[-\frac{(y-y_0)^2}{2r_B^2}\right]H_n\left(\frac{y-y_0}{r_B}\right) \qquad (3)$$

$\omega_B=eB/m_p$, the cyclotron frequency, $r_B=(\hbar/eB)^{1/2}$, the magnetic radius, m_p is the mass of the particle, $H_n(\xi)$ are Hermite polynomials. We omit here spin interaction with magnetic field. Factors $C_n(p_y)$ satisfy the usual condition:

$$\sum_{n=0}^{\infty}|C_n(p_y)|^2=1. \qquad (4)$$

For their definition one should use continuity of the wavefunction at $t=0$:

$$\frac{1}{(2\pi\hbar)^{1/2}}\exp\left\{\frac{i}{\hbar}p_y(y-y_0)\right\}=\sum_{n=0}^{\infty}C_n(p_y)\chi_n(y).$$

Hence, it follows that

$$C_n(p_y)=\frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty}\exp\left(\frac{i}{\hbar}p_y(y-y_0)\right)\chi_n(y)\,dy. \qquad (5)$$

Inserting (3) into (5) and calculating the integral [1], one has

$$C_n(p_y)=\frac{r_B^{1/2}i^n}{\hbar^{1/2}\pi^{1/4}(2^n n!)^{1/2}}\exp\left\{-\frac{1}{2}\left(\frac{r_B}{\hbar}p_y\right)^2\right\}H_n\left(\frac{r_B}{\hbar}p_y\right)=\frac{r_B^{1/2}}{\hbar^{1/2}}i^n\varphi_n\left(\frac{r_B}{\hbar}p_y\right) \qquad (6)$$

with†

$$\varphi_n(\xi)=\frac{1}{\pi^{1/4}(2^n n!)^{1/2}}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi).$$

Since the magnetic field acts perpendicular to its direction, the most interesting case is when x and z components of kinetic momentum are zero. Then $p_y r_B/\hbar=(E/E_0^B)^{1/2}$ (E is the total energy of the particle, $E_0^B=\hbar\omega_B/2$), and

$$C_n^{rd}\equiv\left(\frac{\hbar}{r_B}\right)^{1/2}C_n=i^n\varphi_n\left(\left(\frac{E}{E_0^B}\right)^{1/2}\right). \qquad (7)$$

† At the process of δ-normalization it should be borne in mind that the δ-function $\delta(\xi)$ is measured in units of $[1/\xi]$. This explains the seeming discrepancy between the fact that C_n are dimensionless and their representation in (6).

Probability of finding particle in the nth state is defined by

$$|C_n^{rd}|^2 = \varphi^2\left(\left(\frac{E}{E_0^{\mathrm{B}}}\right)^{1/2}\right). \tag{8}$$

Graphs and tables of functions $\varphi_n(\xi)$ can be found, for instance, in [2].

It is easy to check that the average y value $\langle y\rangle = \int \psi y \psi^* \,\mathrm{d}y$ of the wavepacket (2) obeys the classical equation of the harmonic oscillator

$$\frac{\mathrm{d}^2\langle y\rangle}{\mathrm{d}t^2} + \omega_{\mathrm{B}}^2\langle y\rangle = 0$$

and $\langle(\Delta y)^2\rangle = \langle y^2\rangle - \langle y\rangle^2$ oscillates with frequency $2\omega_{\mathrm{B}}$, as would be expected for arbitrary wavepackets in a uniform magnetic field [3, 4].

We now want to point out the most characteristic features of the $|C_n^{rd}|^2$ dependence on energy E and intensity B, which follow from the properties of functions $\varphi_n(\xi)$ [2]. If $E \ll E_0^{\mathrm{B}}$ (i.e. $E \approx 0$ or (and) $B \to \infty$), then after switching, transitions are possible only to even Landau states, and the probability of finding the particle on the level $2m$ changes with m according to the law $(2m)!/2^{2m}(m!)^2$. Transitions to odd Landau levels are almost completely depressed. In the opposite case, $E \gg E_0^B$ ($E \to \infty$ or (and) $B \approx 0$), transitions are possible only to levels with large numbers, $n \gg 1$, and interaction with low states, because of its very small value, can be neglected. As seen from the graphs of $\varphi_n(\xi)$ [2], for higher E/B, the higher levels come into play. However, on further increase in E/B, interaction with these levels vanishes making way for the states with still larger n. $|C_n^{rd}|^2$ dependence on E and B at their intermediate values is characterized, according to (8), by the square of functions $\varphi_n(\xi)$, which are well known and have been tabled in detail in [2, 5].

It is interesting to investigate $|C_n^{rd}|^2$ in the case $E = mE_0^B$, m integer. For instance, if, before switching, the particle had ground-state energy of magnetic field, $E = E_0^{\mathrm{B}}$, then after applying the field the transition to the first excited state is the most probable process, then to the fourth level, and only after that does the probability of the process at which the particle does not change energy occur.

The results are easily generalized to the movement of wavepackets. If, for instance, at $t < 0$ the wavefunction is

$$\psi(x, y, z, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} b(p_x)b(p_y)b(p_z) \exp\left\{\frac{\mathrm{i}}{\hbar}[p_x x + p_z z + p_y(y - y_0)]\right\}$$
$$\times \exp\left[-\mathrm{i}\frac{p_x^2 + p_y^2 + p_z^2}{2m_p\hbar} t\right] \mathrm{d}p_x \,\mathrm{d}p_y \,\mathrm{d}p_z \tag{9}$$

then after switching one obtains

$$\psi(x, y, z, t) = \frac{1}{2\pi\hbar} \sum_{n=0}^{\infty} C_n \exp\left[-\mathrm{i}\left(n + \frac{1}{2}\right)\omega_{\mathrm{B}} t\right] \int_{-\infty}^{\infty} b(p_x)b(p_z)\chi_n(y)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}[p_x x + p_z z]\right\} \exp\left[-\mathrm{i}\frac{p_z^2}{2m_p\hbar} t\right] \mathrm{d}p_x \,\mathrm{d}p_z. \tag{10}$$

A procedure similar to that described above gives the following expression for C_n:

$$C_n = \int_{-\infty}^{\infty} f_n(p_y)\, \mathrm{d}p_y \tag{11a}$$

$$f_n(p_y) = \frac{1}{(2\pi\hbar)^{1/2}} b(p_y) \int_{-\infty}^{\infty} \exp\left(\frac{\mathrm{i}}{\hbar} p_y(y - y_0)\right) \chi_n(y)\, \mathrm{d}y$$
$$= \left(\frac{r_B}{\hbar}\right)^{1/2} \mathrm{i}^n b(p_y) \varphi_n\left(\frac{r_B}{\hbar} p_y\right). \tag{11b}$$

If the packet is Gaussian with RMS deviation Γ_{p_y}:

$$b(p_y) = \frac{1}{(2\pi)^{1/2}\Gamma_{p_y}} \exp\left\{ -\frac{(p_y - p_y^{(0)})}{2\Gamma_{p_y}^2} \right\} \tag{12}$$

then substituting (12) into (11) and calculating the integral [6] gives:

$$C_n^{rd} = \mathrm{i}^n \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \frac{1}{\sqrt{1+\xi_M^2}} \left(\frac{1-\zeta_M^2}{1+\zeta_M^2}\right)^{n/2} \exp\left\{ -\frac{\zeta_0^2}{2(1+\zeta_M^2)} \right\} H_n\left(\frac{\zeta_0}{1+\zeta_M^2} \left(\frac{1+\zeta_M^2}{1-\zeta_M^2}\right)^{1/2} \right)$$
$$\zeta_M = \frac{r_B \Gamma_{p_y}}{\hbar} \qquad \zeta_0 = \frac{r_B p_y^{(0)}}{\hbar}. \tag{13}$$

At $\Gamma_{p_y} \to 0$, equation (13) transforms to (7) for interaction of the plane wave with momentum $p_y^{(0)}$ with varying field.

Let us now consider the solution of the problem when the magnetic field $\boldsymbol{B}$ is switched off at $t=0$: $\boldsymbol{B}(t) = \boldsymbol{B}h(-t)$. Wavefunctions are, at $t<0$:

$$\psi(x,y,z,t) = \frac{1}{2\pi\hbar} \exp\left\{ \frac{\mathrm{i}}{\hbar}[p_x x + p_z z] \right\} \exp\left(-\mathrm{i}\frac{p_z^2}{2m_p \hbar} t \right)$$
$$\times \exp\left[-\mathrm{i}\left(m + \frac{1}{2}\right) \omega_B t \right] \chi_m(y) \tag{14}$$

(we suppose that the particle is in the mth Landau state); and at $t>0$:

$$\psi(x,y,z,t) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left\{ \frac{\mathrm{i}}{\hbar}[p_x x + p_z z] \right\} \exp\left[-\mathrm{i}\frac{p_x^2 + p_z^2}{2m_p \hbar} t \right]$$
$$\times \int_{-\infty}^{\infty} D_m(p_y) \exp\left[\frac{\mathrm{i}}{\hbar} p_y (y - y_0) \right] \exp\left[-\mathrm{i}\frac{p_y^2}{2m_p\hbar} t \right] \mathrm{d}p_y. \tag{15}$$

Factors $D_m(p_y)$ are

$$D_m(p_y) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{\mathrm{i}}{\hbar} p_y (y - y_0) \right] \chi_m(y)\, \mathrm{d}y = \left(\frac{r_B}{\hbar}\right)^{1/2} (-\mathrm{i})^m \varphi_m\left(\frac{r_B}{\hbar} p_y\right). \tag{16}$$

One can easily check that

$$\int_{-\infty}^{\infty} |D_m(p_y)|^2\, \mathrm{d}p_y = 1 \qquad m = 0, 1, \ldots. \tag{17}$$

As one can see, equation (17) in the switching-off process plays the same role as equation (4) in the switching-on procedure.

In the same way solutions are built for the case of more than one switching. For instance, if the magnetic field is switched on at $t=0$ and switched off at $t=T(\boldsymbol{B}(t)=\boldsymbol{B}[h(t)-h(t-T)])$, then the wavefunction at $t>T$ is:

$$\psi(x,y,z,t)=\frac{1}{(2\pi\hbar)^{3/2}}\exp\left\{\frac{\mathrm{i}}{\hbar}[p_x x+p_z z]\right\}\exp\left[-\mathrm{i}\frac{p_x^2+p_z^2}{2m_p\hbar}t\right]$$
$$\times\int_{-\infty}^{\infty}C(p_y,p_y')\exp\left[\frac{\mathrm{i}}{\hbar}p_y'(y-y_0)\right]\exp\left[-\mathrm{i}\frac{p_y'^2}{2m_p\hbar}t\right]\mathrm{d}p_y' \tag{18}$$

with

$$C(p_y,p_y')=\frac{r_\mathrm{B}}{\hbar}\exp\left[-\mathrm{i}\frac{p_x^2+p_y'^2}{2m_p\hbar}T\right]\sum_{n=0}^{\infty}\varphi_n\left(\frac{r_\mathrm{B}}{\hbar}p_y\right)\varphi_n\left(\frac{r_\mathrm{B}}{\hbar}p_y'\right)$$
$$\times\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\omega_\mathrm{B}T\right]. \tag{19}$$

It follows from (19) that

$$|C(p_y,p_y')|^2=\left(\frac{r_\mathrm{B}}{\hbar}\right)^2\sum_{n=0}^{\infty}\left\{\varphi_n^2\left(\frac{r_\mathrm{B}}{\hbar}p_y\right)\varphi_n^2\left(\frac{r_\mathrm{B}}{\hbar}p_y'\right)+2\varphi_n\left(\frac{r_\mathrm{B}}{\hbar}p_y\right)\varphi_n\left(\frac{r_\mathrm{B}}{\hbar}p_y'\right)\right.$$
$$\left.\times\sum_{l=n+1}^{\infty}\varphi_l\left(\frac{r_\mathrm{B}}{\hbar}p_y\right)\varphi_l\left(\frac{r_\mathrm{B}}{\hbar}p_y'\right)\cos[(l-n)\omega_\mathrm{B}T]\right\} \tag{20}$$

and

$$\int_{-\infty}^{\infty}|C(p_y,p_y'|^2\,\mathrm{d}p_y'=\frac{r_\mathrm{B}}{\hbar}\sum_{n=0}^{\infty}\varphi_n^2\left(\frac{r_\mathrm{B}}{\hbar}p_y\right)=1. \tag{21}$$

We will now consider the situation when the magnetic field is switched off at $t=0$ and switched on again at $t=T(\boldsymbol{B}(t)=\boldsymbol{B}[h(-t)+h(t-T)])$. At $t<0$ the wavefunction is expressed by (14), at $0<t<T$, by (15), and at $t>T$ it is

$$\psi(x,y,z,t)=\frac{1}{2\pi\hbar}\exp\left\{\frac{\mathrm{i}}{\hbar}[p_x x+p_z z]\right\}\exp\left(-\mathrm{i}\frac{p_z^2}{2m_p\hbar}t\right)$$
$$\times\sum_{k=0}^{\infty}C_{km}\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)\omega_\mathrm{B}t\right]\chi_k(y). \tag{22}$$

Matching solutions at $t=0$ and $t=T$, one obtains

$$C_{km}=\mathrm{i}^{k-m}\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)\omega_\mathrm{B}T\right]\exp\left(-\mathrm{i}\frac{p_x^2}{2m_\mathrm{p}\hbar}T\right)I_{km}(\omega_\mathrm{B}T) \tag{23a}$$

$$I_{km}(\xi)=\frac{1}{\pi^{1/2}(2^{m+k}m!k!)^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\frac{\xi}{2}p^2\right)\exp(-p^2)H_k(p)H_m(p)\,\mathrm{d}p. \tag{23b}$$

It is seen from (23), that a particle from an even (odd) level after two switchings can make a transition only to an even (odd) state (i.e. transitions are possible only between levels with numbers $2m \to 2(m+j)$ or $2m+1 \to 2(m+j)+1$, $j=0, \pm 1, \pm 2, \ldots$). Calculation gives [6]:

$$I_{km}(\omega_B T) = \frac{\mathrm{i}^{k-m}}{\pi^{1/2}} \left(\frac{2^{m+k}}{m!k!} \right)^{1/2} \Gamma\left(\frac{m+k+1}{2} \right) \frac{\left(-\mathrm{i} \frac{2}{\omega_B T} \right)^{1/2}}{\left[1 - \mathrm{i} \frac{2}{\omega_B T} \right]^{(m+k+1)/2}} \times {}_2F_1\left(-m, -k; \frac{1-m-k}{2}; \frac{1}{2}\left[1 - \mathrm{i} \frac{2}{\omega_B T} \right] \right) \tag{24a}$$

with $\Gamma(\xi)$ Γ-function and ${}_2F_1(a, b; c; \xi)$ a Gaussian hypergeometric function [5], and

$$C_{km} = \frac{(-1)^k}{\pi^{1/2}} \exp\left[-\mathrm{i}\left(k + \frac{1}{2} \right) \omega_B T \right] \exp\left(-\mathrm{i} \frac{p_x^2}{2m_p \hbar} T \right) \left(\frac{2^{m+k}}{m!k!} \right)^{1/2} \Gamma\left(\frac{m+k+1}{2} \right) \times \frac{\left(-\mathrm{i} \frac{2}{\omega_B T} \right)^{1/2}}{\left[1 - \mathrm{i} \frac{2}{\omega_B T} \right]^{(m+k+1)/2}} {}_2F_1\left(-m, -k; \frac{1-m-k}{2}; \frac{1}{2}\left[1 - \mathrm{i} \frac{2}{\omega_B T} \right] \right) \tag{24b}$$

where $(k+m)$ is even. It is also seen that $|C_{km}|^2$ is symmetric to the replacement of k and m.

Probabilities of transitions from the ground state are

$$|C_{2k,0}|^2 = \frac{(2k)!}{2^{2k}(k!)^2} \frac{2}{\omega_B T} \frac{1}{\left[1 + \left(\frac{2}{\omega_B T} \right)^2 \right]^{k+1/2}} \tag{25a}$$

$$|C_{0,0}|^2 = \frac{1}{\left[1 + \left(\frac{\omega_B T}{2} \right)^2 \right]^{1/2}}. \tag{25b}$$

The analogous values for the first excited state are as follows:

$$|C_{2k+1,1}|^2 = \frac{(2k+1)!}{2^{2k}(k!)^2} \left(\frac{2}{\omega_B T} \right)^3 \frac{1}{\left[1 + \left(\frac{2}{\omega_B T} \right)^2 \right]^{k+3/2}} \tag{26a}$$

$$|C_{1,1}|^2 = \frac{1}{\left[1 + \left(\frac{\omega_B T}{2} \right)^2 \right]^{3/2}}. \tag{26b}$$

One can easily show [7] that

$$\sum_{k=0}^{\infty} |C_{2k+\sigma,\sigma}|^2 = 1 \qquad \sigma = 0, 1. \tag{27}$$

Obviously, equations similar to (27) are true for any other level.

It follows from (25a) that maximal probability for transition from ground state to the $2k$th level is achieved at $\omega_B T = 2\sqrt{2k}$, and for transition from the first excited state to the $(2k+1)$th level (equation (26*a*)) at $\omega_B T = 2\sqrt{\frac{3}{2}k}$.

Increasing the number of switchings, one obtains a periodic process with frequency $\omega = \pi/T$. This process simulates the electron behaviour in a variable magnetic field. It follows from (19), (20) and (24)–(26) that varying ω_B/ω (i.e. changing intensity B or (and) frequency ω) one can change the probabilities of electromagnetic absorption or emission over a wide range. Thus, not only space [8], but also time-varying magnetic fields may be used for the desired electromagnetic generation in radio and light frequencies ranges.

It is instructive to investigate the process in which the magnetic field is suddenly changing from $B_1 \neq 0$ to $B_2 \neq 0$. In this case the initial wave function is

$$\psi(x, y, z, t) = \frac{1}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}[p_x x + p_z z]\right\} \exp\left(-i\frac{p_z^2}{2m_p\hbar}t\right) \times \chi_m^{(1)}(y) \exp\left[-i\left(m+\frac{1}{2}\right)\omega_1 t\right]. \tag{28}$$

After switching ($t>0$) one obtains

$$\psi(x, y, z, t) = \frac{1}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}[p_x x + p_z z]\right\} \exp\left(-i\frac{p_z^2}{2m_p\hbar}t\right) \times \sum_{n=0}^{\infty} C_{nm}\chi_n^{(2)}(y) \exp\left[-i\left(n+\frac{1}{2}\right)\omega_2 t\right]. \tag{29}$$

Here

$$\chi_n^{(i)}(y) = \frac{1}{\pi^{1/4} r_i^{1/2} (2^n n!)^{1/2}} \exp\left[-\frac{(y-y_{0i})^2}{2r_i^2}\right] H_n\left(\frac{y-y_{0i}}{r_i}\right)$$

$$\omega_i = \frac{eB_i}{m_p} \qquad r_i = (\hbar/(eB_i))^{1/2} \qquad y_{0i} = -\frac{p_x}{eB_i} \qquad i = 1, 2.$$

As in the previous cases

$$\sum_{n=0}^{\infty} |C_{nm}|^2 = 1 \qquad m = 0, 1, \ldots. \tag{30}$$

Calculation of C_{nm} gives [6]†

$$C_{nm} = \exp\left[-\frac{1}{2}\frac{y_{01}}{r_1}\frac{y_{02}}{r_2}\frac{(B_2-B_1)^2}{(B_1+B_2)(B_1B_2)^{1/2}}\right]\left(\frac{(B_1B_2)^{1/2}}{(B_1+B_2)2^{n+m-1}n!m!}\right)^{1/2} \times \left(\frac{B_1-B_2}{B_1+B_2}\right)^{m/2}\left(\frac{B_2-B_1}{B_2+B_1}\right)^{n/2}\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}k!\left(1-\frac{B_1}{B_2}\right)^{k/2} \times \left(1-\frac{B_2}{B_1}\right)^{k/2} H_{m-k}\left(-\frac{y_{01}}{r_1}\left(\frac{B_1-B_2}{B_1+B_2}\right)^{1/2}\right) H_{n-k}\left(-\frac{y_{02}}{r_2}\left(\frac{B_2-B_1}{B_2+B_1}\right)^{1/2}\right). \tag{31}$$

† Equation 2.20.16.11 in [6] is wrong; namely, factor '2' should be removed from all radicands, both in denominator (first radicand) and in numerators (radicands of arguments of Hermite polynomials).

In particular, for transitions from the ground state we have

$$C_{n0}=\exp\left[-\frac{1}{2}\frac{y_{01}}{r_1}\frac{y_{02}}{r_2}\frac{(B_2-B_1)^2}{(B_1+B_2)(B_1B_2)^{1/2}}\right]\left(\frac{(B_1B_2)^{1/2}}{(B_1+B_2)2^{n-1}n!}\right)^{1/2}$$

$$\times\begin{cases}\left(\dfrac{B_2-B_1}{B_2+B_1}\right)^{n/2}H_n\left(-\dfrac{y_{02}}{r_2}\left(\dfrac{B_2-B_1}{B_2+B_1}\right)^{1/2}\right) & B_2>B_1\\ (-1)^n\left(\dfrac{B_1-B_2}{B_1+B_2}\right)^{n/2}H_n^*\left(-\dfrac{y_{02}}{r_2}\left(\dfrac{B_1-B_2}{B_1+B_2}\right)^{1/2}\right) & B_2<B_1\end{cases}\tag{32}$$

with $H_n^*(\xi)=(-\mathrm{i})^nH_n(\mathrm{i}\xi)$ [5]. Some properties of functions $H_n^*(\xi)$ are given in the appendix.

For $p_x=0$, equation (31) takes form:

$$C_{nm}(p_x=0)=\left(\frac{(B_1B_2)^{1/2}}{(B_1+B_2)2^{n+m-1}n!m!}\right)^{1/2}\left(\frac{B_1-B_2}{B_1+B_2}\right)^{m/2}\left(\frac{B_2-B_1}{B_2+B_1}\right)^{n/2}$$

$$\times\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}k!\left(1-\frac{B_1}{B_2}\right)^{k/2}\left(1-\frac{B_2}{B_1}\right)^{k/2}H_{m-k}(0)H_{n-k}(0).\tag{33}$$

From (33) and the properties of Hermite polynomials it follows that in this case once again, transitions are possible only between even (odd) states. In particular, for transitions from the ground state

$$C_{2j,0}(p_x=0)=\frac{(-1)^j}{j!}\left(\frac{(B_1B_2)^{1/2}(2j)!}{(B_1+B_2)2^{2j-1}}\right)^{1/2}\left(\frac{B_2-B_1}{B_2+B_1}\right)^j.\tag{34}$$

The probability of remaining in the ground state is

$$|C_{00}(p_x=0)|^2=\frac{2(B_1B_2)^{1/2}}{B_1+B_2}.$$

Using (34) and equation 5.2.13.1 in [7], one can directly show that (30) holds for $m=0$.

After n switchings, which take place after time T, the wavefunction is

$$\psi(x,y,z,t)=\frac{1}{2\pi\hbar}\exp\left\{\frac{\mathrm{i}}{\hbar}[p_xx+p_zz]\right\}\exp\left(-\mathrm{i}\frac{p_z^2}{2m_p\hbar}t\right)$$

$$\times\sum_{k=0}^{\infty}C_{km}^{(n)}\chi_k^{(1)}(y)\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)\omega_1t\right]\qquad n=2j\tag{35a}$$

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$$\times\sum_{k=0}^{\infty}C_{km}^{(n)}\chi_k^{(2)}(y)\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)\omega_2t\right]\qquad n=2j+1\tag{35b}$$

and $C_{km}^{(n)}$ are:

$$C_{km}^{(n)}=\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)(n-1)\omega_1 T\right]\sum_{l_1,l_2,\ldots,l_{n-1}=0}^{\infty}\exp\left[-\mathrm{i}\omega_2 T\sum_{\nu=0}^{j-1}\left(l_{2\nu+1}+\frac{1}{2}\right)\right]$$
$$\times\exp\left[-\mathrm{i}\omega_1 T\sum_{\nu=0}^{j-2}\left(l_{2(\nu+1)}+\frac{1}{2}\right)\right]$$
$$\times\prod_{\nu=0}^{j-1}C_{l_{2\nu+1},l_{2\nu}}C_{l_{2\nu+1},l_{2\nu+2}}\qquad n=2j\qquad j\geqslant 1 \tag{36a}$$

(at $n=2$ the term

$$\exp\left[-\mathrm{i}\omega_1 T\sum_{\nu=0}\left(l_{2(\nu+1)}+\frac{1}{2}\right)\right]$$

vanishes) and

$$C_{km}^{(n)}=\exp\left[-\mathrm{i}\left(k+\frac{1}{2}\right)(n-1)\omega_2 T\right]\sum_{l_1,l_2,\ldots,l_{n-1}=0}^{\infty}\exp\left[-\mathrm{i}\omega_2 T\sum_{\nu=0}^{j-1}\left(l_{2\nu+1}+\frac{1}{2}\right)\right]$$
$$\times\exp\left[-\mathrm{i}\omega_1 T\sum_{\nu=0}^{j-1}\left(l_{2(\nu+1)}+\frac{1}{2}\right)\right]C_{l_1,m}$$
$$\times\prod_{\nu=0}^{j-1}C_{l_{2\nu+1},l_{2\nu+2}}C_{l_{2\nu+3},l_{2\nu+2}}\qquad n=2j+1\qquad j\geqslant 1 \tag{36b}$$

where we assume that $l_0=m$, $l_n=k$, and C_{l_i,l_j} are expressed by equation (31). The probability of finding the particle in the kth state is defined by $|C_{km}^{(n)}|^2$. In particular, after two switchings

$$|C_{km}^{(2)}|^2=\sum_{n=0}^{\infty}C_{nm}C_{nk}\left\{C_{nm}C_{nk}+2\sum_{l=n+1}^{\infty}C_{lm}C_{lk}\cos[(l-n)\omega_2 T]\right\}. \tag{37}$$

As a final example, we consider the situation with $B_1\equiv -B_2\equiv B$. The wavefunction at $t<0$ is expressed by (14), and at $t>0$ is

$$\psi(x,y,z,t)=\frac{1}{2\pi\hbar}\exp\left\{\frac{\mathrm{i}}{\hbar}[p_x x+p_z z]\right\}\exp\left(-\mathrm{i}\frac{p_z^2}{2m_{\mathrm{p}}\hbar}t\right)$$
$$\times\sum_{n=0}^{\infty}C_{nm}\chi_n^{(+)}(y)\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\omega_{\mathrm{B}}t\right] \tag{38}$$

with

$$\chi_n^{(+)}(y)=\frac{1}{\pi^{1/4}r_{\mathrm{B}}^{1/2}(2^n n!)^{1/2}}\exp\left[-\frac{(y+y_0)^2}{2r_{\mathrm{B}}^2}\right]H_n\left(\frac{y+y_0}{r_{\mathrm{B}}}\right). \tag{39}$$

The C_{nm} are [6]:

$$C_{nm}=\exp\left[-\frac{y_0^2}{r_B^2}\right]\begin{cases}\left(2^{n-m}\dfrac{m!}{n!}\right)^{1/2}\left(-\dfrac{y_0}{r_B}\right)^{n-m}L_m^{n-m}\left(2\dfrac{y_0^2}{r_B^2}\right) & n\geqslant m\\ \left(2^{m-n}\dfrac{n!}{m!}\right)^{1/2}\left(\dfrac{y_0}{r_B}\right)^{m-n}L_n^{m-n}\left(2\dfrac{y_0^2}{r_B^2}\right) & n\geqslant m\end{cases} \tag{40}$$

where $L_n^m(\xi)$ are Laguerre polynomials. It is seen from (40) that at $p_x=0$ there are no transitions $C_{nm}(p_x=0)=\delta_{nm}$. From a physical point of view this is explained by degeneracy of the Landau states at $p_x=0$ with respect to the field direction†. At $p_x\neq 0$ the centre of the parabolic well is suddenly switched from point $y=y_0$ to $y=-y_0$, and, thus, transitions are possible. Equations for several switchings are similar to (36).

Two final remarks. First, there are no difficulties in generalization of the results obtained for the simultaneous influence of crossed electric and magnetic fields. Second, the proposed procedure may be used in calculations for all systems with time-varying magnetic fields, for instance, the time-dependent Aharonov–Bohm effect [9] or a potential well in time-varying external electric and magnetic fields—a problem we have solved very recently [10].

In conclusion, we have investigated theoretically for the first time, electron interaction with a magnetic field which suddenly changes in time, and have defined its characteristic features. The properties revealed offer interesting applications in the generation of electromagnetic radiation.

Appendix

We here point out some properties of polynomials $H_n^*(x)$. For comparison similar properties of Hermite polynomials are also given.

$$H_n^*(x)=(-\mathrm{i})^n H_n(\mathrm{i}x) \qquad\qquad H_n(x)=(-\mathrm{i})^n H_n^*(\mathrm{i}x)$$

$$H_n^*(x)=n!\sum_{k=0}^{[n/2]}\frac{(2x)^{n-2k}}{k!(n-2k)!} \qquad\qquad H_n(x)=n!\sum_{k=0}^{[n/2]}\frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}$$

$$H_n^*(x)=\mathrm{e}^{-x^2}\frac{\mathrm{d}^n\mathrm{e}^{x^2}}{\mathrm{d}x^n} \qquad\qquad H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n\mathrm{e}^{-x^2}}{\mathrm{d}x^n}$$

$$H_{n+1}^*(x)=2xH_n^*(x)+2nH_{n-1}^*(x) \qquad\qquad H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$$

$$H_n^{*\prime\prime}(x)+2xH_n^{*\prime}(x)-2nH_n^*(x)=0 \qquad\qquad H_n''(x)-2xH_n'(x)+2nH_n(x)=0$$

$$H_n^{*\prime}(x)=2nH_{n-1}^*(x) \qquad\qquad H_n'(x)=2nH_{n-1}(x)$$

$$\exp(t^2-2\xi t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}H_n^*(\xi)t^n \qquad\qquad \exp(-t^2+2\xi t)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(\xi)t^n$$

$$[\mathrm{e}^{x^2/2}H_n^*]''-[2n+1+x^2][\mathrm{e}^{x^2/2}H_n^*]=0 \qquad\qquad [\mathrm{e}^{-x^2/2}H_n]''+[2n+1-x^2][\mathrm{e}^{-x^2/2}H_n]=0$$

$$H_n^*(x)=\sum_{k=0}^{n}A_{kn}H_k(x) \qquad A_{kn}=\binom{n}{k}2^{n-k}(n-k-1)!! \qquad (n-k)\text{ is even.}$$

† Taking into account interaction with spin removes this degeneracy.

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